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63 (FY)SEM-3/MAJ/MATMAJ2014

2024

MATHEMATICS

Paper : MATMAJ2014

(Elements of Real Analysis)

Full Marks : 70

Pass Marks : 28

Time : 3 hours

The figures in the margin indicate full marks for the questions.

1. Choose the correct answer : $1\times6=6$

(a) The set N of Natural numbers is

(i) bounded above

(ii) not bounded above

(iii) bounded below

(iv) not bounded below

(b) Between two distinct real numbers, there always lies

(i) a rational number

(ii) two rational numbers

(iii) finitely many rational numbers

(iv) infinitely many rational numbers

(c) A series Σu_n is called absolutely convergent if

(i) the series $\Sigma |u_n|$ is devergent

(ii) the series Σu_n is divergent

(iii) the series $\Sigma |u_n|$ is convergent

(iv) the series Σu_n is convergent

(d) If X and Y are countable sets then $X \cap Y$ is also

(i) Countable Set.

(ii) Uncountable Set.

(iii) Both countable and uncountable Set.

(iv) Absolutely uncountable Set.

(e) If a and b are any two positive real numbers such that $a<b$ then there exists a positive integer n such that

(i) $na > nb$

(ii) $na < nb$

(iii) $na \leq nb$

(iv) $na \geq nb$

(f) A positive term series $\sum \frac{1}{n^P}$ is convergent if and only if

(i) $P \geq 1$

(ii) $P < 1$

(iii) $P > 1$

(iv) $P \leq 1$

2. Answer the following questions : ***(any five)*** 2×5=10

(a) Find the supremum and infimum for the set

$$X = \left\{ \frac{1}{n} / n \in N \right\}$$

(b) Prove that a non-empty finite set is not a neighbourhood of any point.

(c) Prove that the sequence $\{a_n\}$ is bounded where

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}, n \in N$$

(d) Show that one series whose n^{th} term is $\sin\frac{1}{n}$ is divergent.

(e) Prove that the sequence $\{x_n\}$ where $x_n = \frac{1}{n}$ is a Cauchy sequence.

(f) Define Cauchy's root test for the convergence of a series.

(g) Show tht the series $\sum \frac{\lfloor n}{n^n}$ is convergent.

3. Answer the following questions : *(any six)* 5×6=30

(a) State and prove Bolzano-Weierstrass theorem for a sequence. 1+4=5

(b) Test the series for convergence

$$1+\frac{x}{\lfloor 1}+\frac{x_2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\cdots\infty$$

(c) If a sequence of closed intervals $[a_n, b_n]$ is such that each member $[a_{n+1}, b_{n+1}]$ is contained in the preceding one $[a_n, b_n]$ and $\lim(b_n - a_n) = 0$. Prove that there is one and only one point common to all the intervals of the sequence.

(d) Show that sequence $\{s_n\}$ where $s_n = \left(1+\frac{1}{n}\right)^n$ is convergent and that $\lim\left(1+\frac{1}{n}\right)^n$ lies between 2 and 3.

(e) Is the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ is conditionally convergent? Justify your answer.

(f) Show that the set of all rational numbers Q is not complete ordered set.

(g) Test the convergence of the following series by Cauchy root test :

$$\sum\left(1+\frac{1}{n}\right)^{-n^2}$$

(h) State Squeeze theorem. Use it to show that

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

2+3=5

(i) Show that the sequence

$x_1=\sqrt{2},\ x_{n+1}=\sqrt{2x_n}$ converges to 2.

4. Answer the following questions : ***(any two)*** 12×2=24

(a) If Σu_n is a positive term series such that

$\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=l$, then prove that the series

(i) converges if $l<1$

(ii) diverges if $l>1$

(iii) the test fails if $l=1$ 5+5+2=12

(b) (i) Prove that a necessary and sufficient condition for a monotonic sequence to be convergent is that it is bounded. 8

(ii) Show that the series

$$\frac{1}{(\log 2)^p}+\frac{1}{(\log 3)^p}+\cdots+\frac{1}{(\log n)^p}+\cdots$$

diverges for $p>0$ 4

(c) (i) If Σu_n is a positive term series such that

$\lim\limits_{n\to\infty}(u_n)^{\frac{1}{n}}=l$, then prove that the series

(i) Converges if $l<1$

(ii) diverges if $l>1$ 4+4=8

(ii) If $\lim a_n=a$ and $a_n\geq 0$ for all n, then prove that $a\geq 0$ 4

(d) (i) If x_1, x_2 are positive and

$x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$ then prove that the two sequences with values

$x_1, x_3, x_5, \cdots$ and $x_2, x_4, x_6, \ldots$

One is decreasing and the other is increasing both converge to the same limit $\frac{1}{3}(x_1 + 2x_2)$ 8

(ii) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent. 4
