

Total number of printed pages – 7

63 (FY)SEM-3/MAJ/MATMAJ2014

2025

MATHEMATICS

Paper : MATMAJ2014

(Elements of Real Analysis)

Full Marks : 70

Pass Marks : 28

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Choose the correct answer : $1 \times 6 = 6$

(a) The set $S = \{x \in \mathbb{R} : x = n + 1, n \in \mathbb{N}\}$ is

(i) Bounded above

(ii) Bounded below

(iii) Bounded

(iv) Unbounded

(b) The infimum of the set

$$\left\{ 1 + \frac{(-1)^n}{n} : n \in \mathbb{N} \right\} \text{ is}$$

(i) 1

(ii) 0

(iii) $\frac{3}{2}$

(iv) does not exist

(c) If a and b are any two positive real numbers such that $a < b$, then there exist a positive integer n such that

(i) $na > nb$

(ii) $na < nb$

(iii) $na \leq nb$

(iv) $na \geq nb$

(d) If $\{x_n\}$ and $\{y_n\}$ are two sequences such that $x_n \leq y_n, \forall n \in \mathbb{N}$. If $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$ then

(i) $x \leq y$

(ii) $y \leq x$

(iii) $x \geq 0$ and $y \leq 0$

(iv) $y \geq 0$ and $x \leq 0$

(e) Which one of the following is correct?

(i) A monotonic increasing sequence is bounded

(ii) A monotonic decreasing sequence is bounded

(iii) A monotonic increasing sequence is unbounded

(iv) A monotonic increasing sequence may or may not be bounded

(f) If $\sum u_n$ is a positive series, such that

$$\lim_{n \rightarrow \infty} \left(n \log \frac{u_n}{u_{n+1}} \right) = l, \text{ then the series}$$

(i) Converges for $l > 1$, and diverges for $l \leq 1$

(ii) Converges for $l \geq 1$, and diverges for $l < 1$

(iii) Converges for $l > 1$, and diverges for $l < 1$

(iv) Converges for $l < 1$, and diverges for $l > 1$

2. Answer the following questions : **(any five)**
2×5=10

(a) Prove that if $a \neq 0$ and b in \mathbb{R} are such that $a \cdot b = 1$, then $b = 1/a$ and if $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

(b) If $a, b \in \mathbb{R}$, then show that
 $|a + b| \leq |a| + |b|$.

(c) Let $a \in \mathbb{R}$. If x belongs to the neighbourhood of $V_\epsilon(a)$ for every $\epsilon > 0$, then show that $x = a$.

(d) Use the definition of limit of a sequence to establish the following limit.

$$\lim \left(\frac{n}{n^2 + 1} \right) = 0$$

(e) State D'Alembert's ratio test for convergence of a series.

(f) Prove that every Cauchy sequence of real numbers is bounded.

(g) Show that the series $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \dots$ is convergent.

3. Answer the following questions : **(any six)**
5×6=30

(a) Prove that if $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x < n_x$.

(b) If x and y are real numbers with $x < y$, then show that there exist a rational number $r \in \mathbb{Q}$ such that $x < r < y$.

(c) For all $a, b, c \in \mathbb{R}$ show that $|ab| = |a||b|$ and if $c \geq 0$, then $|a| \leq c$ if and only if $-c \leq a \leq c$.

(d) Show that every convergent sequence of real numbers is bounded, but the converse is not true.

(e) Test the series for convergence

$$\frac{1.2}{3^2.4^2} + \frac{3.4}{5^2.6^2} + \frac{5.6}{7^2.8^2} + \dots$$

(f) Find all $x \in \mathbb{R}$ that satisfy the following inequality: $|x| + |x+1| < 2$.

(g) Show that for any fixed value of x , the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is convergent.

(h) Show that the sequence $\{S_n\}$, where

$$S_n = \frac{1}{n+1} + \frac{1}{n+1} + \dots + \frac{1}{n+n}, \forall n \in \mathbb{N}$$

is convergent

(i) Show that the series $\sum \frac{1}{n^p}$ diverges if $p \leq 1$.

4. Answer the following questions: **(any two)**
12×2=24

(a) (i) If $I_n = [a_n, b_n], n \in \mathbb{N}$, is a nested sequence of closed bounded intervals, then show that there exists a number $\xi \in \mathbb{R}$ such that $\xi \in I_n$ for all $n \in \mathbb{N}$. 6

(ii) Show that the set \mathbb{R} of real numbers is not countable. 6

(b) If $\sum u_n$ is a positive series, such that

$$\lim_{n \rightarrow \infty} (u_n)^{1/n} = l, \text{ then prove that the series}$$

$$\sum u_n$$

(i) Converges, if $l < 1$,

(ii) Diverges, if $l > 1$, and

(iii) The test fails to give any definite information, if $l = 1$.

$$5+5+2=12$$

(c) State and prove Squeeze theorem. Applying Squeeze theorem, prove that

$$\lim_{n \rightarrow \infty} \left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(2n)^2} \right) = 0.$$

$$6+6=12$$

(d) (i) Define monotone sequence. Prove that a monotone sequence of real numbers is convergent if and only if it is bounded. 1+5=6

(ii) Prove that a bounded sequence of real numbers has a convergent subsequence. 6