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63/1 (SEM-2) DSC/GE2/MATRC2026/MATHG2026

2025

MATHEMATICS

Paper : MATRC2026/MATHG2026

(Algebra)

Full Marks : 80

Pass Marks : 32

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Choose the correct answer of the following : ***(any six)*** 1×6=6

   (a) Which of the following forms a group ?

   (i) $(\mathbb{R}, \cdot)$

   (ii) $(\mathbb{Q}, \cdot)$

   (iii) $(\mathbb{Z}, \cdot)$

   (iv) $(Q \setminus \{0\}, \cdot)$

(b) $(Q,+)$ is a subgroup of

(i) $(\mathbb{Z},+)$

(ii) $(\mathbb{Z},\cdot)$

(iii) $(\mathbb{R},+)$

(iv) $(\mathbb{R},\cdot)$

(c) If x, y are two elements of a ring R then which of the following is true?

(i) $xy = yx$

(ii) $x+y=y+x$

(iii) $xy=0 \Rightarrow x=0$ *or* $y=0$

(iv) $xy \neq 0 \Rightarrow x \neq 0, y \neq 0$

(d) If S is a subring of a ring R then which of the following is false?

(i) S is commutative if R is commutative.

(ii) char R = char S.

(iii) unity element of S and R are same if exist.

(iv) unity element of S and R are not same if exist.

(e) A commutative ring R without zero divisor is

(i) a division ring

(ii) an integral domain

(iii) a field

(iv) a Boolean ring

(f) The order of the symmetric group S_b is

(i) 120

(ii) 720

(iii) 6

(iv) 240

(g) The number of generators in the group $(\mathbb{Z},+)$ is

(i) 1

(ii) 2

(iii) 3

(iv) 4

(h) Which statement is correct of the following :

(i) Every division ring is a field.

(ii) Every integral domain is a field.

(iii) Every commutative ring is a field.

(iv) Every finite integral domain is a field.

(i) A subgroup H of a group G is a normal subgroup if and only if

(i) $Hx = xH \forall x \in G$

(ii) $hx = xh, \forall x \in G, h \in H$

(iii) $Hx \neq xH \forall x \in G$

(iv) $hx \neq xh, \forall x \in G, h \in H$

(j) Let G be a group and a nonempty set $C = \{x \in G : xy = yx \forall y \in G\}$. Then C is called

(i) normaliser

(ii) centraliser

(iii) centre

(iv) commutator of the group

2. Answer ***any five*** of the following questions : 2×5=10

(a) Show that identity element of a group is unique.

(b) If f = (12)(3)(45) and g = (153)(24) are two elements of S_5, then find $f \circ g$.

(c) Prove that every cyclic group is abelian.

(d) Show that in an integral domain D $ab = ac \Rightarrow b = c$ if $a \neq 0$.

(e) Let R be a ring and $x^2 = x \forall x \in R$, prove that R is a commutative ring.

(f) If H is a subgroup of a group G, then show that inverse of 'a' in G and inverse of 'a' in H are same.

(g) Define left and right coset of a subgroup in a group.

3. Answer **any six** of the following questions : $5 \times 6 = 30$

(a) If 'a' is an element of a group G, then prove that its normaliser $N(a) = \{x \in G \mid ax = xa\}$ is a subgroup of G.

(b) Show that a subgroup H of a group G is a normal subgroup iff $xH_x^{-1} = H, \forall x \in G$.

(c) Let H be a subgroup of a group G. Prove that any two right cosets of H in G are either disjoint or identical.

(d) Prove that the disjoint cycles of a permutation commute.

(e) Prove that the intersection of two subgroups is again a subgroup. Is the union of two subgroups a subgroup ? Justify your answer.

(f) If in a ring R with $(xy)^2 = x^2y^2$ for all $x, y \in R$ then show that R is commutative.

(g) Let $R = \{(a_{ij})_{2\times 2} : a_{ij} \in \mathbb{Z}\}$ be a ring of 2×2 matrices over integers and let

$$A = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{Z} \right\}$$

show that A is a right ideal of R but not left ideal of R.

(h) Let G be a group and let 'a' be an element of order n in G. If $a^K = e$. Then prove that n divides K.

(i) Prove that a non-zero finite integral domain is a field.

(j) Show that the set

$$M = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} : a, b, \in \mathbb{R} \right\}$$

is a ring under matrix addition and multiplication.

4. Answer ***any two*** of the following :

10×2=20

(a) Let Q_1 be the set of all rational numbers other than 1. Show that $(Q_1, *)$ is a group where operation is defined by

$$a * b = a + b - ab \forall a, b \in Q_1$$

(b) (i) The set of all cosets $\frac{G}{H} = \{Ha : a \in G\}$ of a normal subgroup H of a group G under the composition of product of cosets defined as $(Ha)(Hb) = Hab \forall a, b \in G$ is a group. 5

(ii) Let $G = \{1, w, w^2\}$, where w is the cube root unity. Construct a Cayley table for the elements of G under the composition of multiplication. Also show that G is a group under the composition of multiplication. 2½+2½=5

(c) Define a cyclic group with an example. Prove that every subgroup of a cyclic group is cyclic. 4+6=10

(d) (i) Show that the set of matrices $\left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b, \in \mathbb{R}\right\}$ is a field under matrix addition and multiplication.

(ii) Show that $R = \left\{a + b\sqrt{2} : a, b, \in \mathbb{Z}\right\}$ is an integral domain. 5+5=10

5. Answer ***any one*** of the following : 14×1=14

(a) (i) Show that the set $\mathbb{R}$ of all real numbers is a commutative ring with unity under ordinary addition and multiplication. 8

(ii) Let R be a commutative ring such that $a^2 = a$ for all $a \in R$. Then Prove that

$(a+b-ab)^2 = a+b-ab$ and

$(a-ab)^2 = a-ab$ for all $a, b, \in R$.

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(b) (i) Let $f = \begin{pmatrix} 1 & 2 & 3 & 5 & 6 & 7 \\ 3 & 5 & 6 & 7 & 1 & 2 \end{pmatrix}$ and $g = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 7 & 6 & 8 & 2 & 3 \end{pmatrix}$ be two permutations on $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Find $f \circ g$, $g \circ f$. Express f, g, $f \circ g$ and $g \circ f$ as a product of disjoint cycles. 2+2+1+1+1+1=8

(ii) If H, K, are two subgroups of a group G then show that HK is a subgoup of G if and only if $HK=KH$. 6

(c) (i) Show that the set

$A_\alpha = \left\{ \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} : \alpha \in \mathbb{R} \right\}$ is a group $w.r.t$ matrix multiplication. Is the group abelian ? Justify your answers. 7+3=10

(ii) Define a commutative ring, an integral domain, a field and a ring with zero divisors. 4