

Total number of printed pages-7

63(FY) SEM-4/MAJ/MATMAJ2034

2025

MATHEMATICS

(Major)

Paper : MATMAJ2034

(Group Theory)

Full Marks : 70

Pass Marks : 28

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Choose the correct answer : $1 \times 6 = 6$

(a) If 'a' and 'b' are any two elements of a group G then $(ab^{-1})^{-1}$ is

(i) $a^{-1}b^{-1}$

(ii) ba^{-1}

(iii) $a^{-1}b$

(iv) $b^{-1}a^{-1}$

(b) The number of generators of an infinite cyclic group is

(i) 1

(ii) 2

(iii) 0

(iv) infinite

(c) If G be a group of order n and H be a proper subgroup of G of order m then

(i) $m = n$

(ii) $m = nq$, q is the index of H in G

(iii) $n = mq$, q is the index of H in G .

(iv) n divides m

(d) A mapping f from a group G into a group G' is said to be homomorphism of G into G' if

(i) $f(ab) = f(a) \cdot f(b)$

(ii) $f(ab) = \frac{f(a)}{f(b)}$

(iii) $f(ab) = f(a) - f(b)$

(iv) $f(ab) = f(a) + f(b)$, $a, b \in G$

(e) If G is a group, then the quotient group G/H is defined if

(i) H is a subgroup of G

(ii) H is a finite subgroup of G

(iii) H is a normal subgroup of G

(iv) G is a subgroup of H

(f) In a group G , if $a^2 = e$, $\forall a \in G$, then G is

(i) Cyclic

(ii) Non-abelian

(iii) Abelian

(iv) Infinite

2. Answer the following question : **(any five)**

$2 \times 5 = 10$

(a) Define Dihedral group. How many elements are there in D_5 ?

(b) Show that $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 1 & 2 & 6 & 7 & 5 \end{pmatrix}$ is an odd permutation.

(c) Prove that in a group G , the identity element is unique.

(d) If $f: G \rightarrow G'$ be homomorphism and e, e' are identity elements of G and G' , respectively, then prove that $f(a^n) = [f(a)]^n$, $a \in G$.

(e) Give reason why any Abelian group of order 15 is cyclic?

(f) Show that every group has at least two normal subgroups.

(g) Justify or invalidate the following statement with argument : "A group of order 24 does possess a normal subgroup of order 14".

3. Answer the following questions : **(any six)**

$$5 \times 6 = 30$$

(a) State Lagrange's theorem an order of a subgroup of a finite group. Prove that

(i) A group of prime order is cyclic.

(ii) For any element 'a' of a group of order n, $a^n = e$.

$$1+2+2=5$$

(b) What do you mean by permutation group? let $S = \{1, 2, 3\}$. Prove that the set S_3 of all the permutations on S is a group under multiplication of permutations. Is S_3 Abelian? Justify your answer.

$$1+3+1=5$$

(c) Define a normal subgroup of a group and give an example. Prove that every subgroup of an abelian group is normal.

$$1+1+3=5$$

(d) Define a homomorphism from a group G to a group G'. Let G be a group and e be the identity element of G. Let $f: G \rightarrow G$ be defined by $f(x) = e, \forall x \in G$. Prove that f is a homomorphism. Is f isomorphism?

$$1+3+1=5$$

(e) If G is a non-commutative group with centre Z, then show that the quotient group G/Z is non-cyclic.

(f) State and prove Cayley's theorem.

$$1+4=5$$

(g) Define left and right cosets of a group. If H is a subgroup of G, then prove that there is a one to one correspondence between set of left coset of H in G and the set of right coset of H in G.

$$2+3=5$$

(h) Define a subgroup of a group. Prove that a non-empty subset H of a group G is a subgroup of G if and only if

$$a, b \in H \Rightarrow ab^{-1} \in H$$

$$1+4=5$$

(i) If H be a normal subgroup of a group G and K, a normal subgroup of G containing H, then show that

$$G/K \cong \frac{G/H}{K/H}$$

4. Answer the following questions : **(any two)**
 $12 \times 2 = 24$

(a) Define automorphism and inner automorphism of a group. If $Z(G)$, $Inn(G)$ and $Aut(G)$ are respectively centre of G , inner automorphism of G and automorphism of G , then show that

$$Inn(G) \trianglelefteq Aut(G) \text{ and } \frac{Aut(G)}{Z(G)} \cong Inn(G)$$

$$2 + 5 + 5 = 12$$

(b) State and prove Cauchy's theorem for Abelian group.

If G is a finite group such that $O(G) = p^2$. Where p is a prime, then prove that G is abelian.

$$2 + 5 + 5 = 12$$

(c) (i) Define a group.

$$\text{Let } G = \left\{ \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} : \alpha \in \mathbb{R} \right\}$$

Prove that G is a group under matrix multiplication.

$$2 + 4 = 6$$

(ii) Prove that the intersection of two subgroups of a group is again a subgroup of the group.

Give an example to show that the union of two subgroups of a group is not a subgroup.

$$4 + 2 = 6$$

(d) (i) If H and K are finite subgroups of G of order $O(H)$ and $O(K)$, respectively then prove that

$$O(HK) = \frac{O(H)O(K)}{O(H \cap K)} \quad 6$$

(ii) Let G be a group such that $(ab)^2 = (ba)^2, \forall a, b \in G$. Suppose G also has the property that $x^2 = e, x \in G \Rightarrow x = e$. Prove that G is abelian.

$$6$$