Inverse of a function

Definition 1 (3.4.1)**.** *Let $f : X \to Y$ be a* ***bijection****. An inverse of f is a map $g : Y \to X$ such that if $f(x) = y$, then $g(y) = x$.*

Why bijectivity is necessary for the existence of an inverse function?
Explanation:

- If f is **not injective**, then two different elements of X may map to the same $y \in Y$. In that case, $g(y)$ would be ambiguous, since it would be unclear which x to choose. Hence, injectivity is necessary to ensure that the inverse, if it exists, is *well-defined*.

- If f is **not surjective**, then there exists some $y \in Y$ for which no $x \in X$ satisfies $f(x) = y$. For such a y, $g(y)$ cannot be defined at all. Hence, surjectivity is necessary to ensure that g is defined for every element of Y.

Therefore, a function f can have an inverse *if and only if* it is a **bijection**.
Uniqueness of Inverse Function.
Suppose $f : X \to Y$ is a bijection. Can f have more than one inverse? The answer is **no**.

If g_1 and g_2 are inverses of f, then they both have domain Y and codomain X. Further, for $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

Then, by the definition of an inverse function, we have

$$g_1(y) = x = g_2(y).$$

Therefore, $g_1 = g_2$.

Hence, we conclude that if f is a bijection, then f has a **unique inverse**.

The inverse of a bijection $f : X \to Y$ is denoted by f^{-1}, which is a function from Y to X.

Exercises

Show that the following functions are bijections and find their inverses.

(a) $f : \mathbb{R} \to \mathbb{R};\ f(x) = 5x - 7$.

(b) $f : [0,1] \to [0,1];\ f(x) = \dfrac{1-x}{1+x}$.

(c) $f : [0,1] \to [a,b]\ (a < b);\ f(x) = a(1-x) + bx$.

(d) $f : [0,\infty) \to [0,1)$, $f(x) = \dfrac{x^2}{1+x^2}$.

(e) $f : [0,\infty) \to [0,1)$; $f(x) = \dfrac{x}{x+1}$

(f) $f : \mathbb{R}^2 \to \mathbb{R}^2;\ f(x,y) = (ax+by,\ cx+dy)$, where $ad - bc \neq 0$.

(g) $f : \mathbb{N} \to \mathbb{N}$;

$$f(m) = \begin{cases} m-1, & \text{if } m \text{ is even,} \\ m+1, & \text{if } m \text{ is odd.} \end{cases}$$

Solutions

(a) $f : \mathbb{R} \to \mathbb{R};\ f(x) = 5x - 7$.

Solution. Injective: Suppose $f(x_1) = f(x_2)$. Then $5x_1 - 7 = 5x_2 - 7$. So $5x_1 = 5x_2$ and $x_1 = x_2$. Thus f is injective.

Surjective: Given $y \in \mathbb{R}$, solve $y = 5x - 7$. Then $x = (y+7)/5 \in \mathbb{R}$. Hence every y has a preimage; f is surjective.

Therefore f is a bijection. The inverse is

$$f^{-1}(y) = \frac{y+7}{5}.$$

(b) $f : [0,1] \to [0,1];\ f(x) = \dfrac{1-x}{1+x}$.

Solution. Injective: Suppose $\dfrac{1-x_1}{1+x_1} = \dfrac{1-x_2}{1+x_2}$. Cross-multiply: $(1-x_1)(1+x_2) = (1-x_2)(1+x_1)$. Simplifying gives $x_1 = x_2$. So f is injective.

Surjective: Given $y \in [0,1]$, solve $y = \frac{1-x}{1+x}$. Rearranging: $y(1+x) = 1-x \Rightarrow x(1+y) = 1-y \Rightarrow x = \frac{1-y}{1+y}$. This x lies in $[0,1]$ because for $y \in [0,1]$, denominator $1+y > 0$ and $0 \le 1-y \le 1$, so $0 \le x \le 1$. . Therefore f is surjective.

Hence f is bijective and the inverse is

$$f^{-1}(y) = \frac{1-y}{1+y}.$$

(c) $f : [0,1] \to [a,b]$ $(a < b)$; $f(x) = a(1-x) + bx$.

Solution. Injective. Suppose $f(x_1) = f(x_2)$. Then

$$a(1-x_1) + bx_1 = a(1-x_2) + bx_2.$$

Expand and collect terms:

$$a - ax_1 + bx_1 = a - ax_2 + bx_2 \quad\Longrightarrow\quad (b-a)x_1 = (b-a)x_2.$$

Since $b - a \neq 0$, we cancel it to obtain $x_1 = x_2$. Hence f is injective.

Surjective. Let $y \in [a,b]$. Solve $y = a(1-x) + bx$ for x:

$$y = a + (b-a)x \quad\Longrightarrow\quad x = \frac{y-a}{b-a}.$$

Because $a \le y \le b \implies 0 \le y-a \le b-a$, we have $0 \le \frac{y-a}{b-a} \le 1$, so the solution $x \in [0,1]$. Thus every $y \in [a,b]$ has a preimage in $[0,1]$, so f is surjective.

Therefore f is a bijection and

$$f^{-1}(y) = \frac{y-a}{b-a}, \qquad y \in [a,b].$$

(d) $f : [0,\infty) \to [0,1)$, $f(x) = \frac{x^2}{1+x^2}$.

Solution. Injective. Suppose $f(x_1) = f(x_2)$. Then

$$\frac{x_1^2}{1+x_1^2} = \frac{x_2^2}{1+x_2^2}.$$

Cross-multiply:

$$x_1^2(1 + x_2^2) = x_2^2(1 + x_1^2)$$

which expands to

$$x_1^2 + x_1^2 x_2^2 = x_2^2 + x_1^2 x_2^2.$$

Cancel the common term $x_1^2 x_2^2$ to get $x_1^2 = x_2^2$. Because the domain is $[0, \infty)$, we have $x_1, x_2 \geq 0$, hence $x_1 = x_2$. Thus f is injective.

Surjective. Let $y \in [0, 1)$. Solve

$$y = \frac{x^2}{1 + x^2}.$$

Rearrange:

$$y(1 + x^2) = x^2 \quad \Longrightarrow \quad y + yx^2 = x^2 \quad \Longrightarrow \quad x^2(1 - y) = y.$$

Since $y \in [0, 1)$, we have $1 - y > 0$ and

$$x^2 = \frac{y}{1 - y} \quad \Longrightarrow \quad x = \sqrt{\frac{y}{1 - y}}.$$

This x is nonnegative, so $x \in [0, \infty)$. For $y = 0$ this gives $x = 0$. Hence every $y \in [0, 1)$ has a preimage in $[0, \infty)$, proving surjectivity.

Therefore f is a bijection and its inverse is

$$f^{-1}(y) = \sqrt{\frac{y}{1 - y}}, \qquad y \in [0, 1).$$

(e) $f : [0, \infty) \to [0, 1); f(x) = \dfrac{x}{x + 1}$

Solution. Injective. Suppose $f(x_1) = f(x_2)$. Then

$$\frac{x_1}{x_1 + 1} = \frac{x_2}{x_2 + 1}.$$

Cross-multiply:

$$x_1(x_2 + 1) = x_2(x_1 + 1) \quad \Longrightarrow \quad x_1 x_2 + x_1 = x_1 x_2 + x_2.$$

Cancel $x_1 x_2$ to obtain $x_1 = x_2$. Thus f is injective.

Surjective. Let $y \in [0, 1)$. Solve

$$y = \frac{x}{x+1}.$$

Rearrange:

$$y(x+1) = x \quad \Longrightarrow \quad yx + y = x \quad \Longrightarrow \quad x(1-y) = y.$$

Since $1 - y > 0$, we get

$$x = \frac{y}{1-y}.$$

This x is nonnegative for $y \in [0, 1)$, so $x \in [0, \infty)$. Thus every $y \in [0, 1)$ has a preimage and f is surjective.

Therefore f is a bijection and its inverse is

$$f^{-1}(y) = \frac{y}{1-y}, \qquad y \in [0, 1).$$

(f) $f(x, y) = (ax + by,\ cx + dy),\ f : \mathbb{R}^2 \to \mathbb{R}^2$, with $ad - bc \neq 0$.

Solution. Injective. Suppose

$$f(x_1, y_1) = f(x_2, y_2),$$

i.e.

$$(ax_1 + by_1,\ cx_1 + dy_1) = (ax_2 + by_2,\ cx_2 + dy_2).$$

Subtract coordinate-wise to obtain

$$ax_1 + by_1 - ax_2 - by_2 = 0, \qquad cx_1 + dy_1 - cx_2 - dy_2 = 0,$$

Simplifying

$$a(x_1 - x_2) + b(y_1 - y_2) = 0, \qquad c(x_1 - x_2) + d(y_1 - y_2) = 0.$$

In matrix form,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0$, the only solution of this homogeneous system is $(x_1 - x_2, y_1 - y_2) = (0, 0)$. Hence $x_1 = x_2$ and $y_1 = y_2$, so f is injective.

Surjective. Let $(u, v) \in \mathbb{R}^2$ be arbitrary. We must find $(x, y) \in \mathbb{R}^2$ with

$$ax + by = u, \qquad cx + dy = v.$$

Because the coefficient matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has nonzero determinant, the linear system has a unique solution given by Cramer's rule:

$$x = \frac{\begin{vmatrix} u & b \\ v & d \end{vmatrix}}{ad - bc} = \frac{ud - bv}{ad - bc}, \qquad y = \frac{\begin{vmatrix} a & u \\ c & v \end{vmatrix}}{ad - bc} = \frac{av - uc}{ad - bc}.$$

These x, y are real, so every (u, v) has a preimage. Thus f is surjective.

Since f is both injective and surjective, it is a bijection. Its inverse is :

$$f^{-1}(u, v) = (\frac{ud - bv}{ad - bc}, \frac{av - uc}{ad - bc}),$$

(g) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(m) = m - 1$ if m is even, and $f(m) = m + 1$ if m is odd.

Solution. Injective

Let $f(m_1) = f(m_2)$. We must show that $m_1 = m_2$.

If m_1 is odd, then $f(m_1) = m_1 + 1$, which is even. Hence $f(m_2)$ is even, so m_2 must also be odd and $f(m_2) = m_2 + 1$. Therefore,

$$m_1 + 1 = m_2 + 1 \implies m_1 = m_2.$$

If m_1 is even, then $f(m_1) = m_1 - 1$, which is odd. Hence $f(m_2)$ is odd, so m_2 must also be even and $f(m_2) = m_2 - 1$. Thus,

$$m_1 - 1 = m_2 - 1 \implies m_1 = m_2.$$

Therefore, f is injective.

Surjective

Let $n \in \mathbb{N}$. We must find $m \in \mathbb{N}$ such that $f(m) = n$.

If n is odd, choose $m = n + 1$ (which is even). Then,

$$f(m) = m - 1 = n.$$

If n is even, choose $m = n - 1$ (which is odd). Then,

$$f(m) = m + 1 = n.$$

Hence, for every $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $f(m) = n$. Therefore, f is surjective.

Since f is both one-one and onto, it is a bijection. Hence, the inverse of f is $f^{-1} = f$.

Remark 2. *The inverse of $f^{-1} : Y \to X$ is $f : X \to Y$, that is, $(f^{-1})^{-1} = f$.*

Proposition 3. *If $f : X \to Y$ is a bijection, then*

$$f^{-1} \circ f = \mathrm{Id}_X \quad \textit{and} \quad f \circ f^{-1} = \mathrm{Id}_Y.$$

Proof. Since f is a bijection, it has a unique inverse $f^{-1} : Y \to X$.

1. For any $x \in X$, let $y = f(x) \in Y$. By definition of inverse,

$$f^{-1}(y) = f^{-1}(f(x)) = x.$$

Thus, for all $x \in X$,

$$(f^{-1} \circ f)(x) = x,$$

which shows $f^{-1} \circ f = \mathrm{Id}_X$.

2. For any $y \in Y$, there exists $x \in X$ such that $f(x) = y$ (because f is surjective). Then

$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = f(x) = y.$$

Hence $f \circ f^{-1} = \mathrm{Id}_Y$.

This completes the proof. ■

Proposition 4. *Suppose $f : X \to Y$ and $g : Y \to X$ are functions such that*

$$g \circ f = \mathrm{Id}_X \quad \textit{and} \quad f \circ g = \mathrm{Id}_Y.$$

Then f is a bijection and $g = f^{-1}$.

Proof. **Injectivity of f:** Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$. Applying g on both sides gives

$$g(f(x_1)) = g(f(x_2)) \implies \mathrm{Id}_X(x_1) = \mathrm{Id}_X(x_2) \implies x_1 = x_2.$$

Hence, f is injective.

Surjectivity of f: Let $y \in Y$. Then

$$y = \mathrm{Id}_Y(y) = (f \circ g)(y) = f(g(y)),$$

so there exists $x = g(y) \in X$ such that $f(x) = y$. Thus f is surjective.

Since f is both injective and surjective, it is a bijection.

Inverse: By definition of inverse, f^{-1} is the unique function such that

$$f^{-1} \circ f = \mathrm{Id}_X \quad \text{and} \quad f \circ f^{-1} = \mathrm{Id}_Y.$$

But g satisfies both $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$. Hence

$$g = f^{-1}.$$

■

Theorem 5. *If $f : X \to Y$ and $g : Y \to Z$ are bijections, then $g \circ f$ is a bijection and*

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Proof. Since f and g are bijections, they have inverses $f^{-1} : Y \to X$ and $g^{-1} : Z \to Y$.

Step 1: Injectivity of $g \circ f$. Suppose $(g \circ f)(x_1) = (g \circ f)(x_2)$ for some $x_1, x_2 \in X$. Then

$$g(f(x_1)) = g(f(x_2)).$$

Since g is injective, it follows that

$$f(x_1) = f(x_2).$$

Since f is injective, we conclude

$$x_1 = x_2.$$

Hence, $g \circ f$ is injective.

Step 2: Surjectivity of $g \circ f$. Let $z \in Z$. Since g is surjective, there exists $y \in Y$ such that $g(y) = z$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. Then

$$(g \circ f)(x) = g(f(x)) = g(y) = z.$$

Hence, $g \circ f$ is surjective.

Step 3: Inverse of $g \circ f$. Consider

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ \mathrm{Id}_Y \circ f = f^{-1} \circ f = \mathrm{Id}_X,$$

and

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ \mathrm{Id}_Y \circ g^{-1} = g \circ g^{-1} = \mathrm{Id}_Z.$$

Thus, by definition of inverse,

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

■

Exercise 0.1. Let $f : X \to Y$ and $g : Y \to Z$ be functions such that $g \circ f$ is surjective and g is injective. Is f surjective?

Solution. Since $g \circ f$ is surjective, so g is surjective. It is also given that g is injective, hence g is bijective and g^{-1} exists. We have, $f = g^{-1} \circ (g \circ f)$. Given that $(g \circ f$ is surjective. Since g^{-1} and $g \circ f$ both are surjective, so $g^{-1} \circ (g \circ f)$ is surjective, that is, f is surjective.