PHYMAJ102-4 (Unit II) <u>Vectors.</u>

Recap

Vector algebra :- There are two operations with vectors -

You know them

i) Vector addition.

ii) Scalar multiplication.

Laws of vector algebra :-

Suppose $\vec{A}, \vec{B}, \vec{C}$ are vectors and "m" and "n" are scalars. Then the following laws holds:

i) $(\vec{A}+\vec{B})+\vec{C} = \vec{A}+(\vec{B}+\vec{C})$

ii) $\vec{A}+\vec{B} = \vec{B}+\vec{A}$

iii) $m(\vec{A}+\vec{B}) = m\vec{A}+m\vec{B}$

iv) $m(n\vec{A}) = (mn)\vec{A}$.

v) $\vec{C}+0 = 0+\vec{C} = \vec{C}$.

Unit vectors :- These are the vectors having unit length. Suppose $\vec{A}$ is any vector with length $|\vec{A}| > 0$. Then $\vec{A}/|\vec{A}|$ is a unit vector, denoted by $\hat{a}$, which has the same direction as $\vec{A}$.

Scalar fields :- If each points in a space P corresponds to a number (scalar) $\phi(x,y,z)$ then $\phi(x,y,z)$ is called a scalar field.

Ex :- $\phi(x,y,z) = xy^2 + zy + 7$.

Vector field :- If each points in a space P corresponds to a vector $\vec{V}(x,y,z)$, then $\vec{V}(x,y,z)$ is called a vector field.

Ex :- $\vec{V}(x,y,z) = xy^2\hat{i} + yz^2\hat{j} + z\hat{k}$.

→ A vector field $\vec{V}$ which is independent of time is called a stationary or steady state vector field.

Dot product

The dot product of two vectors $\vec{A}$ and $\vec{B}$ is defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the cosine of the angle between them.

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta \quad ; \quad 0 \le \theta \le \pi$$

Propositions :-

→ $\vec{A}\cdot\vec{B}$ is a scalar quantity

→ $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ → Commutative law

→ $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$ ⟶ Distributive law

→ If $\vec{A}\cdot\vec{B} = 0$ and $\vec{A}$ and $\vec{B}$ are not null vectors, then $\vec{A}$ and $\vec{B}$ are perpendicular.

Cross product :-

The cross product of two vectors $\vec{A}$ and $\vec{B}$ is a vector $\vec{C} = \vec{A}\times\vec{B}$ is defind as follows. The magnitude of $\vec{C} = \vec{A}\times\vec{B}$ is equal to the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the sine of the angle between them. The direction of $\vec{C} = \vec{A}\times\vec{B}$ is perpendicular to the plane of $\vec{A}$ and $\vec{B}$ such that $\vec{A}$, $\vec{B}$ and $\vec{C}$ forms a right-handed system

$$\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta\,\hat{n} \quad ; \quad 0 \le \theta \le \pi$$

where, $\hat{n}$ is the unit vector indicating the direction of $\vec{C}$ or $\vec{A}\times\vec{B}$.

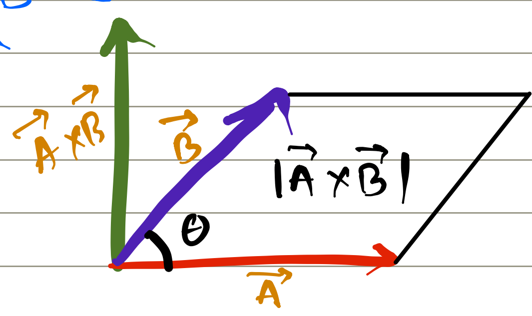
Preposition :-

→ $\vec{A}\times\vec{B} = -(\vec{B}\times\vec{A})$; commutative law

→ $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$; Distributive law.

→ If $\vec{A}\times\vec{B} = 0$ and $\vec{A}$ and $\vec{B}$ are not null vectors then $\vec{A}$ and $\vec{B}$ are parallel.

→ The magnitude of $\vec{A}\times\vec{B}$ is the same as the area of a parallelogram with sides $\vec{A}$ and $\vec{B}$.



Triple products :- → scalar triple product
→ vector triple product

The dot and cross products of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ may produce meaningful products called triple products of the form $(\vec{A}\cdot\vec{B})\vec{C}$, $\vec{A}\cdot(\vec{B}\times\vec{C})$ and $\vec{A}\times(\vec{B}\times\vec{C})$, etc.

Preposition :-

→ In general $(\vec{A}\cdot\vec{B})\,\vec{C} \neq \vec{A}\,(\vec{B}\cdot\vec{C})$

→ In general $(\vec{A}\times\vec{B})\times\vec{C} \neq \vec{A}\times(\vec{B}\times\vec{C})$

Associative law for cross product fails.

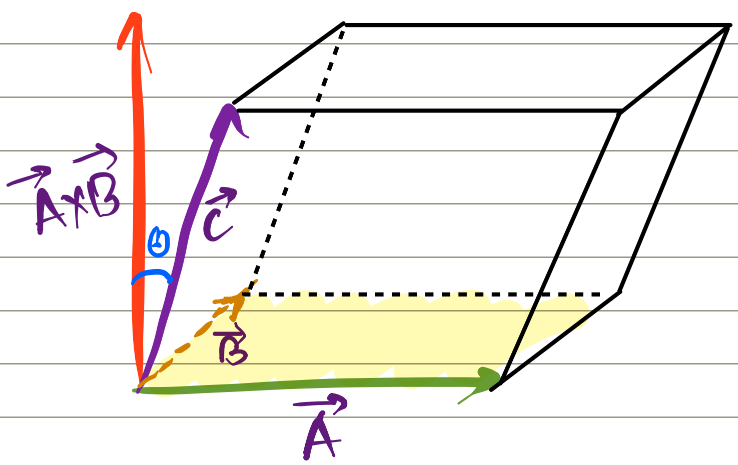
→ $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\,\vec{B} - (\vec{A}\cdot\vec{B})\,\vec{C}$

→ $(\vec{A}\times\vec{B})\times\vec{C} = (\vec{A}\cdot\vec{C})\,\vec{B} - (\vec{B}\cdot\vec{C})\vec{A}$

→ $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$ represents volume of a parallelepiped having $\vec{A}$, $\vec{B}$ and $\vec{C}$ as edges.

Scalar triple product
↳ $\vec{A}\cdot\vec{B}\times\vec{C}$

vector triple product
↳ $\vec{A}\times(\vec{B}\times\vec{C})$

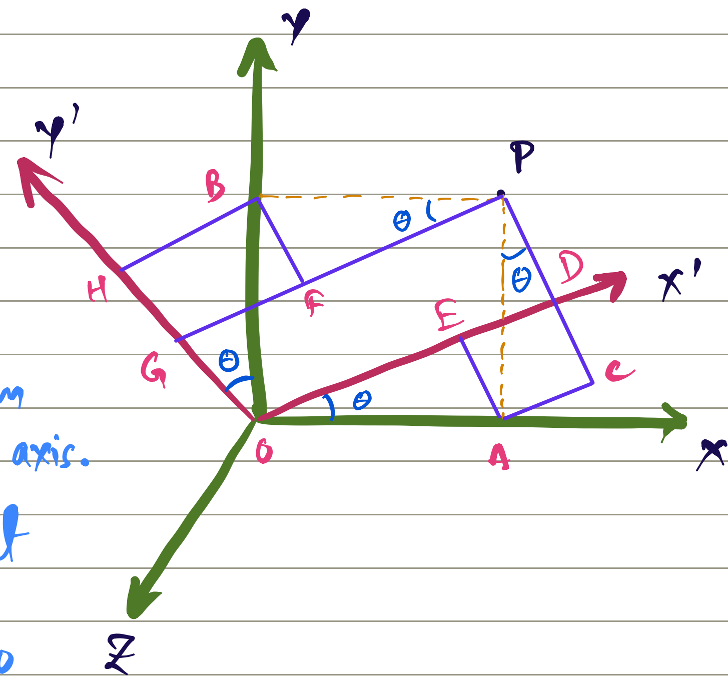


Here the shadded region represents the area of the plane or $|\vec{A}\times\vec{B}|$. The triple product with $\vec{C}$ gives the volume of the parallelepiped, i.e.,

$$|\vec{C}\cdot(\vec{A}\times\vec{B})|.$$

→ $(\vec{A}\cdot\vec{B})\times\vec{C}$ is a meaningless expression. You can't make a cross product from a scalar and a vector.

Rotation of vectors :-



Let us consider a coordinate system (x, y, z) being rotated by an angle "θ" to the new coordinate system (x', y', z') with respect to z-axis.

Suppose there is a point $P(x, y, z)$. Now we need to find the transformed axes with respect to the previous axes.

First we draw PA and PB, two $\perp^r$ lines on x and y-axis respectively. Again we draw PD and PG on the new axes x' and y'. A few more perpendiculars are drawn for our convience.

From $\triangle OAE$,

$$\cos\theta = \frac{OE}{OA} \Rightarrow OE = OA\cos\theta = x\cos\theta$$

Similarly, from $\triangle PAC$,

$$\sin\theta = \frac{AC}{PA} \Rightarrow AC = PA\sin\theta = y\sin\theta$$

$$\therefore OD = OE + ED$$
$$= x\cos\theta + y\sin\theta \quad \Big| \; ED = AC$$
$$\Rightarrow x' = x\cos\theta + y\sin\theta \longrightarrow (1)$$

Now, consider $\triangle OHB$ —

$$\cos\theta = \frac{OH}{OB} \Rightarrow OH = OB\cos\theta = y\cos\theta$$

Similarly from $\triangle PFB$,

$$\sin\theta = \frac{BF}{PB} \Rightarrow BF = PB\sin\theta = x\sin\theta$$

Now, $OG = OH - HG$

$$\Rightarrow OG = y\cos\theta - BF$$

$\Rightarrow y' = y\cos\theta - x\sin\theta \longrightarrow (2)$

As these axes rotates with respect to z axis, therefore it does not changes.

$\therefore z = z'. \longrightarrow (3)$

Now, eq. (1), (2) and (3) can be express in terms of matrix form as —

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \vec{r}' = R(\theta)\vec{r} \quad \#$$

$\Rightarrow$ If we consider a new vector $\vec{A}$ such that

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

Then, $\vec{A}' = R(\theta)\vec{A}$

where,

$$A'_x = A_x\cos\theta + A_y\sin\theta$$

$$A'_y = -A_x\sin\theta + A_y\cos\theta$$

$$\therefore \vec{A}' = A'_x\hat{i} + A'_y\hat{j} + A'_z\hat{k}.$$

$$= (A_x\cos\theta + A_y\sin\theta)\hat{i} + (-A_x\sin\theta + A_y\cos\theta)\hat{j} + A_z\hat{k}$$

Q. Prove that magnitude of a vector under rotation remains invariant.

or

Prove $|\vec{A}| = |\vec{A}'|$

Q. Prove the invariance of dot product of two vectors under rotation.

or

Prove that $\vec{A}\cdot\vec{B} = \vec{A}'\cdot\vec{B}'$.